Optimal Payload Lofting with Tethers

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The problem of optimal use of a tether to increase the final energy of a payload is posed. The optimal solution must consist of zero tension arcs, full tension arcs, and possibly singular arcs. For a massless tether, the state solution can be obtained for both full tension and free arcs, reducing the system to a parameter optimization problem. Optimal trajectories begin with a coasting arc to obtain separation between the payload and the orbiter, and then the payload is accelerated toward the orbiter with positive cable tension. Optimal lofting trajectories thus require reel-in deployment systems. Example solutions are given for several values of the cable tension. The maximum reel-in speed at the winch device is always a substantial fraction of the payload release speed.

Introduction

THE use of tethers between two satellites to propel the smaller was first studied by Colombo.[1] In his method, an initial impulse leads to separation of the payload, and then deployment continues under positive cable tension once the gravity gradient force is strong enough. The tether connecting two objects in different orbits will transfer momentum between the lower (faster) object and the higher (slower) object. In what has been termed a "swing release," the upper object thus gains speed at the expense of the lower satellite and has its orbit raised. This concept has been further explored by Bekey[2] and Bekey and Penzo.[3] This latter review shows a simple cable deployment mechanism using tension on the tether only during the separation phase. While simple to build, since the tether deployment mechanism only needs a drag device, such trajectories are limited to tether tensions less than the gravity gradient force tending to separate the two satellites.

If the tether tension overwhelms the gravity gradient force, the satellites will approach each other. In this paper we consider the case where a high tension tether is used to deploy a smaller payload. We will find that the optimal deployment solutions involve accelerating the payload toward and past the orbiter.

The Optimal Tether Problem

The dynamics of two masses connected by a tether has been extensively studied from the point of view of modeling the dynamics of the tether itself. However, in this paper we shall model the tether as a massless, inextensible string with an upper limit on the tension it may carry without breaking. Figure 1 shows two masses, m_1 (the payload) and m_2 (the orbiter), connected by such a tether. It is well known that the center of mass of the system remains in its initial orbit, which we shall take to be circular with orbital angular rate Ω. The relative coordinates are the vertical separation ρ, and the in-track separation measured from the orbiter to the payload η. Assuming that these remain small, the kinetic and potential energies can be expanded, retaining terms through the second order. Then, in the rotating reference frame of Fig. 1, the Lagrangian for the dynamical system takes the form

$$L = \frac{1}{2} M\left[\dot{\rho}^2 + \dot{\eta}^2 + 2\Omega(\rho\dot{\eta} - \dot{\rho}\eta) + \Omega^2(\rho^2 + \eta^2)\right] - \frac{1}{2} M\,\Omega^2\left(\eta^2 - 2\rho^2\right) \tag{1}$$

where $M = m_1 m_2/(m_1 + m_2)$ is the system-reduced mass. The second line represents the gravity gradient potential. The tension T in the tether will introduce generalized force components into the equations of motion

$$Q_\rho = -\rho T/r, \qquad Q_\eta = -\eta T/r \tag{2}$$

where r is the separation distance between the payload and the orbiter. So, if $x^T = \{\rho, \eta, \dot{\rho}, \dot{\eta}\}$ is the state vector, the equations of motion can be written as $\dot{x} = f(x) + g(x)\tau$ or

$$\dot{x}_1 = x_3, \qquad \dot{x}_2 = x_4$$

$$\dot{x}_3 = 2\Omega x_4 + 3\Omega^2 x_1 - \frac{x_1}{\sqrt{x_1^2 + x_2^2}}\tau$$

$$\dot{x}_4 = -2\Omega x_3 - \frac{x_2}{\sqrt{x_1^2 + x_2^2}}\tau \tag{3}$$

Here, the control variable $\tau = T/M$ is physically the acceleration of the tether produced by its tension.

The problem of deploying a satellite from a larger object, say the Space Shuttle, involves controlling the tension τ in the tether in such a way that given final conditions are realized. In the optimal tether deployment problem, this means that certain combinations of the final states

$$J = \phi[x(t_f)] \tag{4}$$

must be optimized. The optimal control problem is then defined by the control Hamiltonian function

$$H = \lambda^T(f + g\tau) \tag{5}$$

which is linear in the control tension τ. Such problems are treated by Bryson and Ho,[4] who show that there are three possible cases for the control variable. The switching function for this problem is given by the part of H that depends on τ or

$$S = -x_1\lambda_3 - x_2\lambda_4 \tag{6}$$

If $S < 0$, H is maximized when the cable tension is zero. We will call these free arcs. If $S > 0$, then H is maximized when τ is as large as possible. Let the limit on the cable acceleration be

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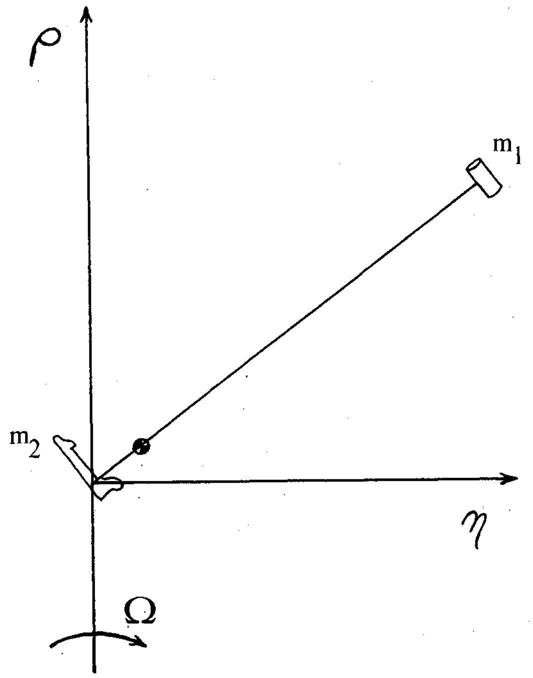

Fig. 1 Rotating orbital coordinate frame for the relative motion of the orbiter and payload.

τ^+. Then, on these arcs, $\tau = \tau^+$ is a constant. Finally, should $S = 0$ for some finite time interval, we would have a singular arc. In this paper we will assume that only free and full tension arcs exist and will reserve the possible existence of singular arcs for future work. Since deployment begins with $\rho = \eta = 0$, the optimal problem must begin with a free arc, since a nonzero tension will simply keep the payload attached to the orbiter.

We will not find it necessary to continue with this formalism any further. Knowing that the optimal solution consists of free arcs and full tension arcs is sufficient to focus our attention on these two special cases. As we shall see in the next section, analytic solutions are obtainable for both of these cases. This reduces the system to a parameter optimization problem, avoiding the use of Lagrange multipliers.

State Solutions

When the satellite does not have tension applied to the tether, the relative motion will occur on the time scale of one orbital period. The state equations of motion

$$\ddot{\rho} - 2\Omega\dot{\eta} - 3\Omega^2\rho = 0$$

$$\ddot{\eta} + 2\Omega\dot{\rho} = 0 \tag{7}$$

form a linear, constant coefficient system. The solution is

$$x(t) = \Phi(t, t_o)\, x(t_o) \tag{8}$$

where x is the state vector given in the previous section. The state transition matrix is given explicitly by

$$\Phi = \begin{Bmatrix} 4-3\cos\psi & 0 & \frac{1}{\Omega}\sin\psi & \frac{2}{\Omega}(1-\cos\psi) \\ 6(\sin\psi-\psi) & 1 & \frac{2}{\Omega}(\cos\psi-1) & \frac{4}{\Omega}\sin\psi-\frac{3}{\Omega}\psi \\ 3\Omega\sin\psi & 0 & \cos\psi & 2\sin\psi \\ 6\Omega(\cos\psi-1) & 0 & -2\sin\psi & 4\cos\psi-3 \end{Bmatrix} \tag{9}$$

where $\psi = \Omega(t - t_o)$ is the phase of the solution.

When the tether is under full tension, the state equations take the form

$$\ddot{r} - r\dot{\theta}^2 - 2\Omega r\dot{\theta} - 3\Omega^2 r\sin^2\theta = -\tau^+$$

$$r\ddot{\theta} + 2\dot{r}(\Omega + \dot{\theta}) - 3\Omega^2 r\sin\theta\cos\theta = 0 \tag{10}$$

in rotating polar coordinates. Now, the terms linear in Ω can be eliminated if we define our polar coordinates with respect to a nonrotating frame. The terms in Ω^2 include centripetal acceleration terms and gravity gradient terms, and we will assume that these latter terms are negligible when the tether is under tension. For a 100 km tether, these terms yield an acceleration less than 0.1 m/s^2, and we shall assume that this can be ignored relative to the tether tension.

The simplified equations of motion are:

$$\ddot{r} - r\dot{\theta}^2 = -\tau^+$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{11}$$

They possess the energy and angular momentum integrals

$$\epsilon = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \tau^+ r \tag{12}$$

$$h = r^2\dot{\theta} \tag{13}$$

Substituting for $\dot{\theta}$ from Eq. (13) into the energy equation and separating variables, we have

$$\int \mathrm{d}t = \int \frac{r\,\mathrm{d}r}{\sqrt{-2\tau^+ r^3 + 2\epsilon r^2 - h^2}} \tag{14}$$

The cubic in the denominator above always has two positive roots, $\alpha > \beta$, the "aposhuttle" and "perishuttle" radii, and one negative root δ. Integrating from $r = \beta$, the perishuttle radius, the left integral above becomes $t - T_o$, where T_o is the time of perishuttle passage. The integral on the right then leads to the equation for the orbit in terms of the auxiliary angle ϕ:

$$r = \frac{\beta - m\,\delta\sin^2\phi}{1 - m\sin^2\phi} \tag{15}$$

and the analog of Kepler's equation for ϕ

$$t - T_o = c_1\, E(\phi\,|\,m) + c_2\frac{\sin 2\phi}{\Delta(\phi)} + c_3\, F(\phi\,|\,m) \tag{16}$$

Here F and E are the incomplete elliptic integrals of the first and second kind, their parameter is $m = (\alpha - \beta)/(\alpha - \delta)$ and

$$\Delta(\phi) = \sqrt{1 - m\sin^2\phi} \tag{17}$$

The constant c_i are

$$c_1 = \frac{\sqrt{2}\,(\beta - \delta)}{\sqrt{\tau^+(\alpha - \delta)}\,(1 - m)}$$

$$c_2 = -\frac{m}{2}c_1$$

$$c_3 = \frac{\sqrt{2}\,\delta}{\sqrt{\tau^+(\alpha - \delta)}} \tag{18}$$

Since Eq. (15) shows that the period of the orbit is π in the modular angle ϕ, the time period of the relative motion is given by

$$T = 2c_1\, E(m) + 2c_3\, K(m) \tag{19}$$

where K and E are the complete elliptic integrals of the first and second kind. This is the period of the relative trajectory in time, perishuttle to perishuttle. As Fig. 2 shows, the trajectory does not close in space.

Returning to Eq. (14), and using the angular momentum law to eliminate time in favor of θ, we find a pair of integrals that yield

$$\theta = \theta_o + \frac{\sqrt{2}}{\sqrt{\tau^+(\alpha-\delta)}\,\beta\delta}\Big\{\beta F(\phi|m) + (\delta-\beta)\,\Pi(\delta m/\beta;\ \phi|m)\Big\} \tag{20}$$

where $\Pi(n;\phi|m)$ is the incomplete elliptic integral of the third kind. Here θ_o may be termed the argument of perishuttle.

The solution can now be completed by differentiation. The velocity components are given by

$$\dot{r} = \frac{m(\beta-\delta)}{\Delta^4(\phi)} \sin 2\phi\ \dot{\phi}$$

$$\dot{\theta} = h/r^2 \tag{21}$$

The rate of change of the auxiliary angle ϕ, also needed if a Newton-Raphson technique is used to solve Eq. (16), is

$$\frac{dt}{d\phi} = c_1\Delta + c_2\left(\frac{2\cos 2\phi}{\Delta} + \frac{m\sin^2 2\phi}{2\Delta^3}\right) + \frac{c_3}{\Delta} \tag{22}$$

The "orbital elements" of the relative trajectory may be taken as ϵ, h, T_o, and θ_o. Solving for the elements given a set of initial conditions begins with the evaluation of ϵ and h from Eqs. (12) and (13). The roots of the cubic equation in Eq. (14) are obtained next, enabling the evaluation of the auxiliary constants c_i. Finally, T_o and θ_o are found from Eqs. (16) and (20), respectively. Solving for the state given the time begins with the iterative solution of Eq. (16) for ϕ. An initial guess for ϕ can be based on the trajectory period and the knowledge that $0<\phi<\pi$, with $\phi=0$ at perishuttle. Knowing ϕ, the state follows directly from Eqs. (15), (20), and (21). The elliptic integrals are easily and efficiently calculated by the process of the arithmetic/geometric mean, as detailed in Abramowitz and Segun.[5] The polar coordinate position and velocity calculated from this solution is then converted to rectangular coordinates and transformed into the rotating frame used for the free arc solution. The full tension solution is then correct, including terms in Ω from Eq. (10), but neglecting the Ω^2 terms arising from the gravity gradient potential.

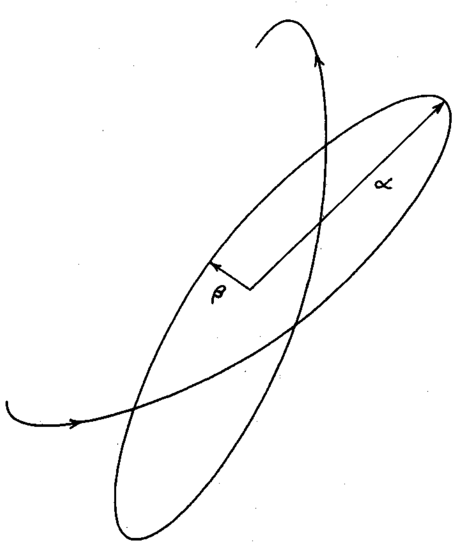

Fig. 2 Sample trajectory of the payload relative to the orbiter when the tether is under high tension. The aposhuttle and perishuttle radii α and β are shown.

The Optimization Problem

With the possession of the state solutions, we have the ability to propagate any set of initial conditions forward in time, through any sequence of free and full tension arcs. The initial position components are obviously $\rho = \eta = 0$, while the initial velocity components are unknown. However, it makes no sense to attempt to maximize the payload energy if the initial ejection speed is a free variable, since cannon-like solutions will be the result. Instead, hold the ejection speed v_o constant, and write the initial ejection velocity components as

$$\dot{\rho}_o = -v_o \sin\theta_o, \qquad \dot{\eta}_o = -v_o\cos\theta_o \tag{23}$$

where the free parameter θ_o can be used in the optimization process. Also unknown are the switching times T_{sw} at which the cable tension is turned on or off. It is also necessary to be more definite as to what is being optimized in this trajectory problem. We have chosen to maximize the two-body orbital energy per unit mass of the payload at the release point. This can be written

$$E = \frac{1}{2}\left((\dot{\rho}-\Omega\eta)^2 + (V_o + \dot{\eta} + \Omega\rho)^2\right) - \frac{\mu}{R_o+\rho} \tag{24}$$

where V_o and R_o are the speed and radius in the initial circular orbit, and μ is the gravitational constant. Because the relative coordinates are small, this can be expanded to second order in the position components, giving the energy increase as

$$\Delta E = \tfrac{1}{2}(\dot{\rho}-\Omega\eta)^2 + V_o(\dot{\eta}+\Omega\rho) + \tfrac{1}{2}(\dot{\eta}+\Omega\rho)^2 + \frac{\mu\rho}{R_o^2} - \frac{\mu\rho^2}{R_o^3} \tag{25}$$

Using the state solutions, we can calculate this as a function of the initial ejection angle θ_o and the switching time T_{sw}. Implicitly then, $\Delta E = \Delta E(\theta_o, T_{sw})$, a function of two constants, and we have a simple parameter optimization problem.

For a simple trajectory with one free arc and one full tension arc, the conditions to be met at the final time are

$$\frac{\partial}{\partial\theta_o}\Delta E = 0, \qquad \frac{\partial}{\partial T_{sw}}\Delta E = 0 \tag{26}$$

Rather than attempting to calculate these derivatives analytically, they have been obtained numerically. Furthermore, if the matrix of second partial derivatives is also obtained numerically, it is simple to set up a gradient scheme to find the optimal θ_o and T_{sw} and to check the type of extremum found.

However, it is useful to have some approximate optimal solution as a starting point for the numerical calculations. If the cable length is kept very short, implying short final time T_f, and the cable tension is high, then the problem can be worked as if the system were in gravity-free space. The outgoing zero tension arc is then the straight line

$$r = v_o t \tag{27}$$

The full tension arc will also be a straight line passing through the orbiter

$$r(t) = -v_o t + \tfrac{1}{2}\tau^+(t-T_{sw})^2 \tag{28}$$

which has been matched to the outgoing trajectory at the switching time. To maximize the kinetic energy of the payload, full tension should be continued until the payload comes through the orbiter, at $r=0$. Solving for the switching time gives

$$T_{sw} = T_f - \sqrt{\frac{2v_oT_f}{\tau^+}} \tag{29}$$

where T_f is the final (release) time. This solution will be a maximum in the orbital energy, Eq. (25), if the ejection occurs rearward along the orbit ($\theta_o = 0$), and the payload is winched forward. Ejection in the forward direction followed by acceleration to the rear will produce a minimum in ΔE.

Results

This optimization problem has been studied for deployment of a small payload from an orbiter in a 90-min period orbit. If the mass of the payload is small, then the reduced mass of the system approximates the mass of the payload itself, and we do not have to specify the exact payload/orbiter mass ratio. Solutions were found in the vicinity of the expected reel-back solution for small T_f as discussed in the previous section, and these were followed numerically toward longer final times T_f. Although the optimization is done for a fixed final time T_f, we have varied T_f to obtain complete families of solutions.

Figure 3 shows several such trajectories for various final times. The ejection velocity from the orbiter was $v_o = 10$ m/s, and the limiting cable acceleration was $\tau = 2$ m/s^2. Note that the vertical scale of the figure has been stretched by a factor of 2. The actual trajectories are local maxima in the energy increase through final times of 90 min and beyond. They maintain the fundamental character of the approximate solution discussed earlier, in that high payload velocity is generated by reeling the payload in at a constant acceleration. The dots on the curves mark the switching points between the free arcs and the constant tension arcs. The payload almost immediately turns around once tension is applied, and most of the remainder of the curve is an inward acceleration toward the orbiter. The optimal deorbit solutions are close to being reflections of these solutions through the origin. Instead of ejecting the payload upward and slightly forward (for large T_f) to obtain a large displacement behind the orbiter, the optimal deorbit solutions obtain a large displacement in front of the orbiter by ejection downward and slightly to the rear, and then winch the payload backward.

Figure 4 shows several of the speeds involved in this problem as a function of the final release time T_f. The top curve in the figure gives the payload release speed in kilometers per second. It is closely tracked by the curve below, which is the maximum velocity of the cable measured at the winch. Since the maximum value of dr/dt occurs when $d^2r/dt^2 = 0$, Eqs. (10) and (13) show that the maximum winch speed occurs when

$$r = \sqrt[3]{h^2/\tau^+} \tag{30}$$

For all of the cases studied, this is always an *inward* velocity. For this low-tension case, it approaches 0.5 km/s as a maximum, and for higher tension cases it can be much higher. However, these are extreme values, and Fig. 4 shows many solutions within the capabilities of motor and drum winch systems. For high-speed cases it is probably not desirable to design the winch to actually wind up the cable. Rather, the cable would be allowed to proceed out the other side of the orbiter. Since cable just exiting the orbiter has higher speed than that which has preceded it, the new cable should drag the entire mass clear of the orbiter. The final curve is the magnitude of the cable velocity at release. This curve is much smaller in magnitude but varies in sign between negative values for small T_f through a positive range and back to negative values. It is desirable to release the payload by severing the cable at the payload, since this does not encumber the payload with the large amount of trailing mass. However, caution would have to be exercised by the orbiter, since there is no method by which the orbiter can decelerate the mass of incoming cable once the payload is released. This would essentially mandate using trajectories where the final cable velocity dr/dt is positive, unless suboptimal trajectories are considered. By adding the side condition $dr/dt(T_f) = 0$ with a Lagrange multiplier to the optimization, Eq. (26), this difficulty can be avoided.

Figure 5 shows the apogee radius of the resulting payload orbits for three values of the cable tension limit τ^+. All of these cases still used ejection velocities v_o of 10 m/s. The higher tension cases, as expected, show higher maximum altitudes. In no case did any payload orbit intersect the Earth, in spite of the fact that a tangency condition $[d\rho/dt(T_f) = 0]$ at release was not included in the optimization problem. Figure 6 shows that the time the cable spends under acceleration $T_f - T_{sw}$ also decreases with increasing cable tension, as expected.

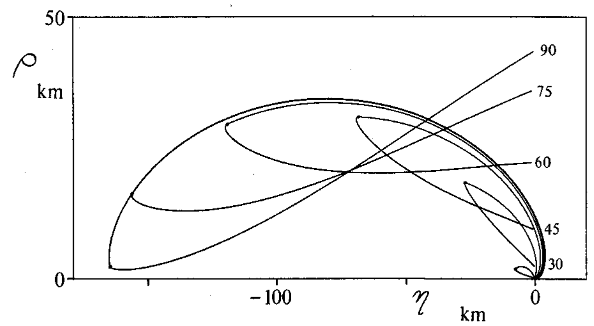

Fig. 3 Optimal deployment trajectories for $v_o = 10$ m/s and $\tau^+ = 2$ m/s^2. Trajectories are shown for 15-min intervals in the final time T_f. The vertical scale is double the horizontal. Dots indicate the switching points on the trajectories.

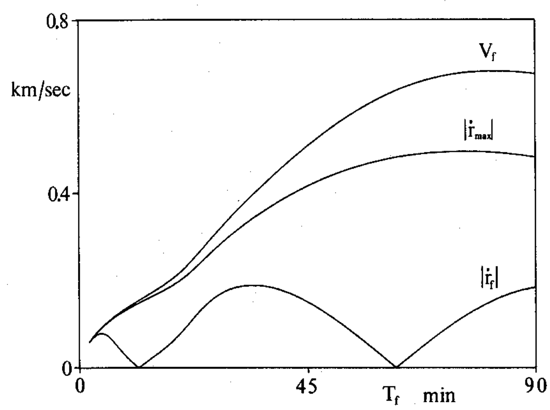

Fig. 4 Release speed V_f, maximum reel-in speed $|\dot{r}_{max}|$, and cable speed at release $|\dot{r}_f|$. The maximum inward cable speed is always a large fraction of the release speed.

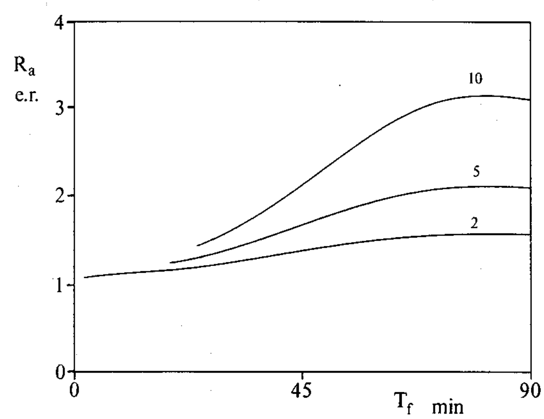

Fig. 5 Apogee radius R_a in Earth radii vs final time T_f for three cases with cable accelerations of $\tau^+ = 2$, 5, and 10 m/s^2.

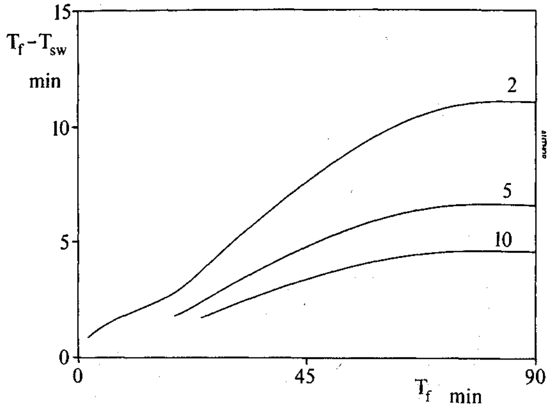

Fig. 6 Time under acceleration $T_f - T_{sw}$ vs final time T_f for the same three cases shown in the previous figure.

Discussion

We have seen that the optimal deployment trajectories for high-tension tethers involve the acceleration of the payload during reel-in of the tether. The deployment mechanism thus must be designed for constant tether tension and high-speed cable retrieval. Such a deployment mechanism would, of course, be more complex than the drag-only deployment systems using "swing release" trajectories. Perhaps using the cable itself as part of a linear electric motor would make possible the high reel-in speeds demanded by the optimal trajectory solution.

These results show that long maximum cable lengths are, as expected, the result of either T_{sw} or v_o, or both being large. In order to keep the total cable length short, it is necessary to keep T_{sw} short. Then, to obtain high release velocities, large tension τ^+ is required. This may be the best route to a practical tether deployment system. However, this essentially returns us to the straight line solution, where the payload hits the orbiter at the release time. In this case, a suboptimal solution with a specified payload-orbiter separation at the final time would be preferred.

The fact that the system center of mass stays in the original orbit has important consequences for the practical use of tethers to loft payloads. *The same amount of linear momentum received by the payload is also deposited in the orbiter.* If the Space Shuttle is the orbiter, an unwanted deorbit is the certain result of lofting a large payload into a high orbit. Using the Shuttle's maneuvering engines to remain in orbit results in only marginal performance improvement over putting solid engines on the payload. This small advantage would almost certainly vanish if the mass of the cable system is considered. However, this difficulty is eliminated if a large amount of unneeded mass can be used as the orbiter. A Space Shuttle external tank or the last stage of an expendable booster are two possibilities, in which case the tether might be powered by excess propellants left in the tanks. Also, the use of a tether waste disposal system to reboost the space station might eliminate the need to supply maneuvering fuel to that vehicle. The required speed change of ≈ 100 m/s for this application should not prove a challenge to winch technology.

Extension of this approach to multiple free and full tension arcs is relatively simple. The release energy, Eq. (25), can be calculated after any assumed sequence of arcs and implicitly becomes a function of θ_o and $2N-1$ switching times for a sequence involving N full tension arcs. The cable lengths required for these solutions appear to be unrealistically large, however, so they have not been extensively investigated.

Of course, the least defensible assumption in this study is that the cable is massless. This work needs to be extended to the case of finite tether mass. The stability of a massive tether also needs to be investigated. However, the high-tension deployment case, with its potentially very high release velocities, shows considerable promise.

Conclusions

We have shown that optimal payload lofting trajectories consist of a sequence of free and full tension arcs. State solutions for both of these cases are obtained, assuming that the tether mass was zero and that the tension was always much larger than the gravity gradient force in the tether. Optimal solutions obtain most of the final payload release speed by reeling the payload in, and payload release speed is closely followed by the maximum reel-in speed as seen at the winch device. High release speeds thus require a high-speed winch device. Assured safe separation distances at release and zero cable speed at release can be obtained by adding side conditions to the fundamental optimization problem.

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